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Optimal Equalization of Wideband Coaxial Cable Channels Using "Bump" Equalizers

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Two methods are described for the optimal equalization of a channel with "Bump" Equalizers composed of several adjustable-gain Bode Networks. The first method is a general one and applies a steepest descent algorithm which minimizes the total mean-squared error (MSE) of the equalized channel. It requires continuous gradient information on the error-gain relationship in order to determine exactly the optimum equalizer adjustments and involves a relatively complicated procedure to calculate the gradient. However, the second method, which also applies a steepest descent algorithm, develops the necessary gradient information with knowledge of the error signal only at selected frequencies across the bandwidth occupied by the channel. Under idealized assumptions, it is shown that the gradients obtained by the second method are exact. When the assumptions do not apply exactly, it is shown by computer simulation that the difference between the gradients obtained by the two methods is very small. A significant potential advantage of the second method lies in the hardware realization which only requires the measurement of the channel error at $2M - 1$ frequencies at the equalizing station (where M is the number of Bode Networks in the equalizer). From these frequency domain errors, the gradients can be generated as real-time signals and applied to the appropriate adjustable elements to obtain the optimum gain settings for minimum MSE.

I. INTRODUCTION

The ideal communication channel exhibits a constant input-output gain characteristic over the entire transmission band. In the case of a 3/8-inch coaxial cable system, the cable loss varies from 4 dB/mile at 1 MHz, to 30 dB/mile at 60 MHz; and to compensate the cable loss, repeaters are required at periodic intervals. As is well known, the cascaded repeaters cannot exactly compensate the cable loss and this

mismatch results in the so-called "misalignment" of the cable system. In addition to the mismatch which is present initially, the channel misalignment is affected by the seasonal temperature variation and the aging of the components in the system. The objective of the main cable equalizers is that, after the equalization, the total input-output gain characteristic of the system should be, at all times, as nearly zero dB as possible over the entire message band.

Since the transmission of amplitude information is of major importance in analog coaxial cable systems, various schemes of input-output "amplitude-only" equalization have been studied in the past.[1,2] It should be noted that the analog signal in the coaxial cable channel may contain either voice or digital information, and the transmission of voice-type information can be accomplished without phase equalization. For the transmission of high-speed digital information, the necessary phase equalization is usually furnished in the digital terminals and not in the main coaxial cable path.

In this paper, the Bump Equalizer,[2] which is an "amplitude-only" equalizer, is studied and a new adjustment method is presented. The Bump Equalizer is composed of a number of Bode Networks,[3] each of which can be controlled independently without affecting the other networks in the set (Fig. 1). A typical Bode Network is shown in Fig. 2a and its transfer function has the form of a bump shape in the frequency domain. In the Bump Equalizer of Fig. 3, several Bode Networks are

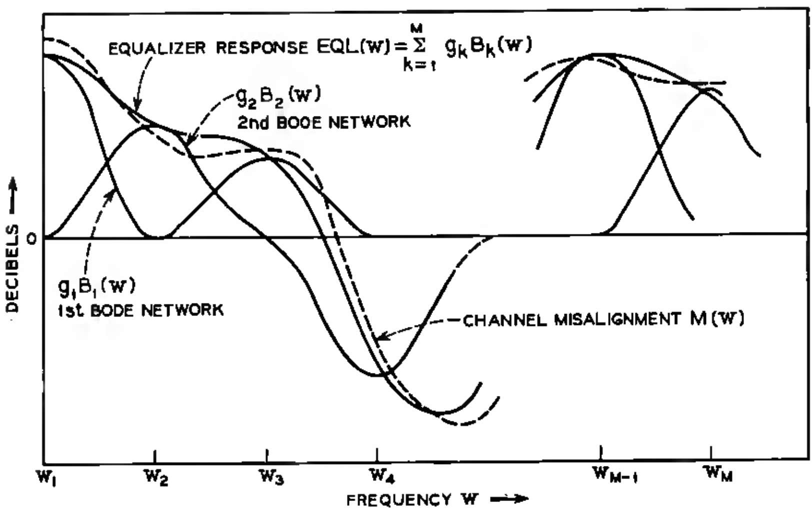

Fig. 1—Channel equalization with Bump Equalizer.

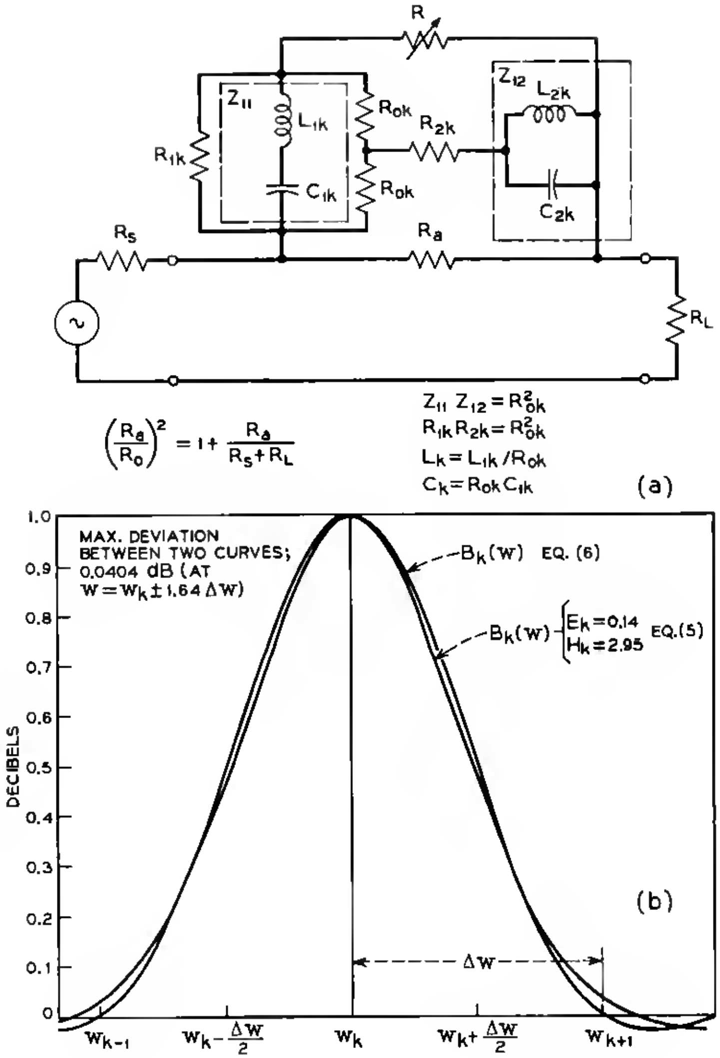

Fig. 2—Bode Network and its input-output transfer function.

connected in the feedback and feedforward paths of linear wideband amplifiers and the transfer function of the equalizer can be expressed by

$$\mathrm{EQL}\,(w) = \sum_{k=1}^{M} g_k B_k(w) \quad (\mathrm{dB}), \tag{1}$$

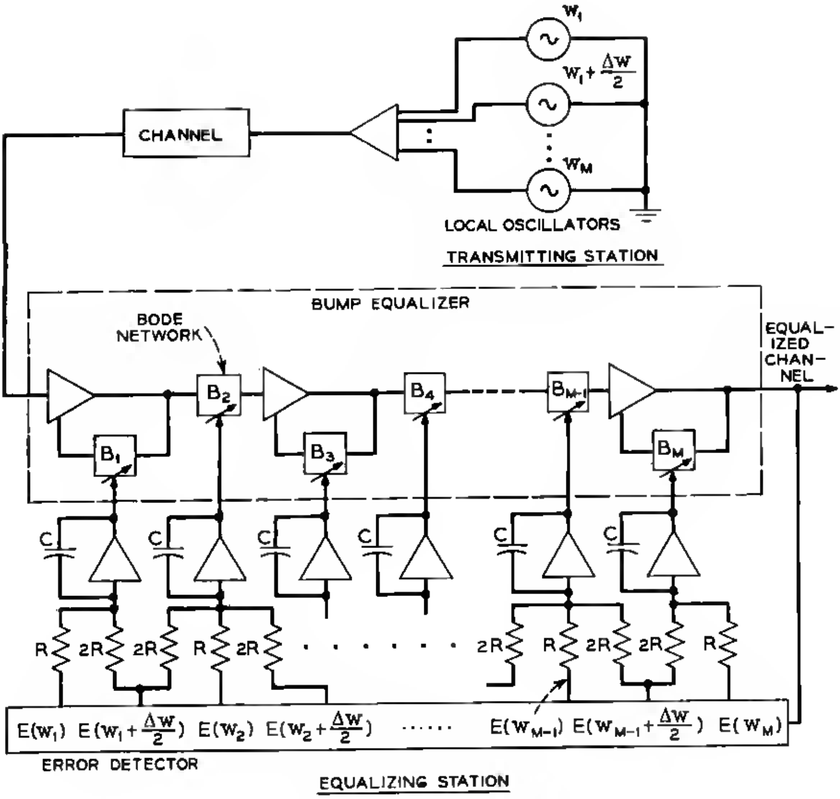

Fig. 3—Equalizer block diagram.

where w indicates the frequency, g_k and B_k represent the gain and response of the kth Bode Network respectively, and M is the number of Bode Networks in the Bump Equalizer.

In the past, the so-called "Zero-Forcing" (ZF) method has been used for the adjustment of Bump Equalizers.[2] Although the ZF method results in zero error at the center frequency of each Bode Network, relatively large errors may exist at other frequencies, and as a result the ZF method may not be optimal in any overall sense. A better error criterion is to minimize the mean-squared error (MSE) over the entire bandwidth, and it is this error criterion which is used in this paper for equalizer gain adjustment.

In Section II, the channel and the transfer function of the equalizer

are characterized, and several assumptions are made before deriving the algorithms which will minimize the MSE of the equalized channel. Before the steepest descent method[4] is used to obtain the minimum MSE, it is discussed in Section III that the MSE of the equalizer has a unique minimum point (no local minima) in the gain parameter space of the equalizer, so that this method is assured of finding the true minimum MSE at all times. It is shown that under certain conditions, the gradient of the MSE, with respect to each gain, can be obtained as a real-time signal by measuring the error at only three points in the frequency domain. The gradient signal derived can be applied to an integrator to adjust the appropriate equalizer gain setting until the gradient becomes zero, and the desired optimization is achieved. This equalization method is called the "simplified MSE" algorithm. Hardware implementation of the algorithm is also discussed in Section III.

Various computer simulations have been carried out and some of the results are discussed in Section IV. Both the conventional expression for the transfer function of a Bode Network and the measured transfer function of a physically realized network have been used in the simulation to verify the effectiveness of the derived algorithm when used in practical applications. One of the channel misalignments used in the simulation resulted from measurements on a working coaxial cable system in the field.

The general steepest descent MSE algorithm is applied to all cases to obtain the absolute minimum MSE for each case; and the resulting values are compared with the MSE obtained by the simplified MSE algorithm. The computer results verify that the simplified algorithm derived under the idealized conditions is, in fact, sufficiently close to the general algorithm in each case so that the former, which permits simplified hardware implementation, can be used as an effective means to achieve optimal control of the Bump Equalizer.

II. CHARACTERIZATION OF CHANNEL AND BUMP EQUALIZER

The coaxial channel is discussed in this paper principally with respect to analog signal transmission. Due to the characteristic of the coaxial cable, the bandwidth of the transmitted signal can be quite wide. The objective of the equalization discussed is to achieve a constant input-output gain characteristic over the entire message band, and the transfer function derived is concerned only with the amplitude characteristic and not the phase characteristic. Since the transfer

function of the Bode Network is symmetric on the "log f" plane, the transformed frequency w, to be used in this section is given by $\log f$ where f is the natural frequency in Hertz.

2.1 *Characterization of Coaxial Communication Channel*

Let $M(w, t)$ represent the time-varying channel misalignment which is a real-valued function of frequency in units of dB. From the practical point of view, however, the channel can be represented as simply $M(w)$, since the time-variance can be assumed negligible during the interval of any equalization process. Assume also that the Fourier transform of the channel misalignment $M(w)$ is limited by a positive constant, since the Bump Equalizer to be used is strictly a frequency domain equalizer.* Hence, the channel can be characterized in the frequency domain by the following series:

$$M(w) = \sum_{n=0}^{\infty} C_n \frac{\sin(2\pi p_N(w - w_n))}{2\pi p_N(w - w_n)} \quad \text{(dB)}, \tag{2}$$

where C_n, p_N, and w_n are certain real numbers, $w = \log f$, and $w_{n+1} - w_n = 1/2p_N$ for all $n = 0, 1, \cdots$.

Eq. (2) also may be expressed as

$$\begin{aligned} M(w) &= \int_0^1 \sum_{n=0}^{\infty} C_n \cos(2\pi p_N(w - w_n)x)\, dx \\ &= \int_0^1 \left\{ \sum_{n=0}^{\infty} C_n \cos(2\pi p_N w_n x) \cos(2\pi p_N w x) \right. \\ &\quad \left. + \sum_{n=0}^{\infty} C_n \sin(2\pi p_N w_n x) \sin(2\pi p_N w x) \right\} dx \\ &= \int_0^1 \{F(x) \cos(2\pi p_N w x) + H(x) \sin(2\pi p_N w x)\}\, dx, \end{aligned} \tag{3}$$

where

$$F(x) = \sum_{n=0}^{\infty} C_n \cos(2\pi p_N w_n x) \quad \text{and} \quad H(x) = \sum_{n=0}^{\infty} C_n \sin(2\pi p_N w_n x).$$

Since $0 \leqq x \leqq 1$, eq. (3) implies that the shortest frequency domain ripple period found in the channel $M(w)$ is $1/p_N$.

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* It should be noted that the Fourier transform of $M(w)$ does not result in an impulse response of the channel because of "dB" dimension of $M(w)$. For this assumption, however, there is an implicit dual relationship with time domain equalizers, e.g., the tapped delay line equalizers, in which a frequency hand limitation of the channel is assumed.

2.2 *Characterization of the Bump Equalizer*

As briefly discussed in the introduction, the frequency domain response of the equalizer can be written as

$$\mathrm{EQL}\,(w) = \sum_{k=1}^{M} g_k B_k(w) \quad (\mathrm{dB}), \tag{4}$$

where M is the number of Bode Networks in the Bump Equalizer.

A typical Bode Network is shown in Fig. 2 and its loss is controlled by the resistor R. The transfer function, $B_k(w)$, can be analytically derived and, with a suitable flat-gain amplifier, can be expressed by the following equation:

$$B_k(w) = \frac{[E_k(1 + E_k) + D_k(w)]^2 - D_k(w)}{[(1 + E_k)^2 + D_k(w)]^2} \quad (\mathrm{dB}), \tag{5}$$

where

$$E_k = \frac{R_{0k}}{R_{1k}},$$

$$D_k(w) = \frac{(w/w_k)H_k}{(w/w_k)^2 - 1},$$

$$H_k = \frac{C_k}{R_{0k}^2 L_k},$$

and

$$w_k = \log\,(1/2\pi\sqrt{L_k C_k}).$$

Since $B_k(w)$ of eq. (5) is a quite complicated function of w, the following assumption is made before analyzing the equalizer in detail.

Assumption 1: Let $B_k(w)$ be approximated by

$$\mathrm{cosinc}\left(\frac{\pi}{\Delta w}(w - w_k)\right) = \frac{\sin \dfrac{\pi}{\Delta w}(w - w_k)}{\dfrac{\pi}{\Delta w}(w - w_k)} \cdot \frac{\cos \dfrac{\pi}{\Delta w}(w - w_k)}{1 - 4\left(\dfrac{w - w_k}{\Delta w}\right)^2} \quad (\mathrm{dB}). \tag{6}$$

Moreover, if there are M Bode Networks in the equalizer, and if they are spaced equally on the w scale at intervals Δw,* such that $\Delta w = w_{k+1} - w_k$ for all $k = 1, \cdots M - 1, M$, then the transfer function of

* Usually the number of Bode Networks, M, is determined from the practical consideration of equalization objectives. p_N defined in eq. (2) determines the Δw which is the interval between two adjacent Bode Networks.

the equalizer can be expressed by

$$\mathrm{EQL}\,(w) = \sum_{k=1}^{M} g_k \,\mathrm{cosinc}\left(\frac{\pi}{\Delta w}\,(w - w_k)\right) \quad (\mathrm{dB}). \tag{7}$$

Equations (5) and (6) are plotted in Fig. 2b; and it can be seen that cosinc $(\pi/\Delta w(w - w_k))$ approximates the actual transfer function of Bode Network given in eq. (5) reasonably well. The maximum difference between the two curves of Fig. 2b is 0.0327 dB when $|w - w_k| \leqq \Delta w$ and 0.0404 dB when $|w - w_k| > \Delta w$.

III. GAIN OPTIMIZATION USING MEAN-SQUARED ERROR CRITERION

After a Bump Equalizer has been physically realized and connected to the channel, the Δw, which is characteristic of the particular set of Bode Networks, cannot be easily altered in the equalizer even though the channel misalignment $M(w)$ (and hence p_N) may vary. The optimization here consists of determining the gain parameters g_k which will minimize the value of MSE defined in this section. One approach to the optimization is the employment of the steepest descent method.[4] In seeking the minimum MSE by this method, the present values of g_k's are changed by small amounts in the opposite direction of gradients which are the partial derivatives of MSE with respect to each gain parameter g_k. The process is continued until all the gradients of the MSE with respect to the gains g_k reach zero or a stationary point. Hence, it is implicit in the use of the steepest descent method that the surface of MSE in the gain parameter space is a bowl shape, and that there exists a unique stationary point which is the global minimum. The unique existence of such a stationary point is established before a general steepest descent algorithm is derived for the Bump Equalizer; then a simplified algorithm is obtained which is shown to be equivalent to the general algorithm. Finally, hardware implementation of the simplified algorithm is discussed.

3.1 *General Mean-Squared Error Algorithm*

On the dB scale, the residual error after equalization will be

$$E(w) = \sum_{k=1}^{M} g_k B_k(w) - M(w) \quad (\mathrm{dB}). \tag{8}$$

If $C(w)$ is the channel characteristic, the channel misalignment is defined by $M(w) = -C(w)$ and the MSE can be represented in the frequency domain by

$$\text{MSE} = \int_{-\infty}^{\infty} E(w)^2 \, dw. \tag{9}$$

It should be noted that the definition in (9) is a bounded functional, since $E(w)$ can be made small as $w > w_M$, where w_M is the upper limit of the band to be equalized and, moreover, there is complete freedom for the assumption of $M(w)$ for $w > w_M$.

Let G_k be a gradient of MSE with respect to gain g_k. Then, G_k is obtained by differentiating MSE with respect to g_k:

$$G_k = \frac{\partial(\text{MSE})}{\partial g_k} = 2 \int_{-\infty}^{\infty} B_k(w)E(w) \, dw. \tag{10}$$

This may be stated as: the gradient G_k with respect to gain g_k is found by cross-correlating the Bode Network function, $B_k(w)$, and the error function, $E(w)$. The cross-correlation method to obtain the gradient in practice has been used elsewhere and can be found in Refs. 5 and 6.

Theorem 1 (General MSE Algorithm): Let $G_{k,j}$ be the gradient of MSE with respect to the kth *gain, g_k, measured at time $t = j$ (also, let $g_{k,j}$ indicate the value of the* kth *gain at time $t = j$). For the Bump Equalizer, the next gain setting of g_k (denoted by $g_{k,j+1}$) which will reduce the MSE is a function of the gain setting $g_{k,j}$ and the gradient $G_{k,j}$, as given by*

$$g_{k,j+1} = g_{k,j} - \Delta c G_{k,j} \tag{11}$$

for all $k = 1, 2 \cdots M$ where Δc is a small positive constant. As the iterative process described by eq. (11) is continued, the gradients $G_k \to 0$ and the equalizer reaches the optimum state.

Proof: Since an equalizer described by (4) is composed of linearly independent networks, there exists a unique set of g_k's which satisfies $G_k = 0$ for all $k = 1, 2, \cdots M$, and this set of g_k's results in the minimum MSE defined in eq. (9) (see Chapter 2 of Ref. 7). Hence, a steepest descent algorithm described by (11) must bring the gains to this optimum stationary point. A general theory on the steepest descent algorithm is given in Ref. 4.

Since no specific assumptions were made on the channel and the Bump Equalizer, the gradient required for the optimum gain adjustment may be obtained by eq. (10).

3.2 *Simplified MSE Algorithm and Hardware Implementation*

In the general steepest descent method described by Theorem 1, the gradient G_k is obtained by the cross-correlation of the error $E(w)$

and the Bode Network function $B_k(w)$. For the hardware implementation of the gradient calculation, the error function is multiplied by the Bode Network response $B_k(w)$ and the product is integrated in the frequency domain. Computation of frequency domain cross-correlation should be done on a real-time basis, and this fact may prohibit the practical application of Theorem 1. However, the following theorem shows that the gradient G_k can be obtained by measuring $E(w)$ only at three different frequencies for each of the given Bode Networks $B_k(w)$.

Theorem 2 (Simplified MSE Algorithm): Let the Bump Equalizer satisfy Assumption 1 and let the interval Δw *between adjacent Bode Networks be no greater than the shortest frequency domain ripple period in the channel shown in (3), i.e.,*

$$\Delta w \leqq \frac{1}{p_N}. \tag{12}$$

Then, the optimum gain setting of the kth *Bode Network is obtained by repeating the following process:*

$$g_{k,j+1} = g_{k,j} - \Delta c\left\{\frac{1}{2} E_j\left(w_k - \frac{\Delta w}{2}\right) + E_j(w_k) + \frac{1}{2} E_j\left(w_k + \frac{\Delta w}{2}\right)\right\}, \tag{13}$$

where

$k = 1, 2, 3, \cdots M$,
Δc is a small positive constant, and
$E_j(w_k - \Delta w/2)$, $E_j(w_k)$, and $E_j(w_k + \Delta w/2)$ are the frequency domain errors measured at time $t = j$ at $w = w_k - \Delta w/2$, $w = w_k$, and $w_k + \Delta w/2$ respectively.

Proof: The proof is given in the Appendix.

In the derivation of the results stated in Theorem 2, it is specified that the channel has a shortest frequency domain ripple period $1/p_N$ which is greater than, or equal to, the interval Δw between two adjacent Bode Networks. If the channel has ripples having components of shorter periods than Δw, then the gradient shown in Theorem 2 only approximates the true gradient which would be obtained by the cross-correlation technique in eq. (10). However, the accuracy of the approximation depends on the amplitude of ripples whose periods are shorter than Δw, and if there exist such ripples of large amplitude, one cannot expect a satisfactory equalization even if a general algorithm is employed.

The feedback equation for the optimum gain adjustment using the simplified MSE technique is as follows:

$$g_k(T) = g_k(0) - \Delta c \cdot \int_0^T \left\{ \frac{1}{2} E\left(w_k - \frac{\Delta w}{2}\right) + E(w_k) + \frac{1}{2} E\left(w_k + \frac{\Delta w}{2}\right) \right\} dt, \tag{14}$$

where $k = 1, 2, 3, \cdots M$, and $g_k(0)$ is an initial value of gain g_k. The hardware implementation of (14) can be achieved in various ways. In the block diagram shown in Fig. 3, the input signal source at the transmitting station is composed of $2M - 1$ unit-amplitude sinusoidal oscillators whose frequencies are w_1, $w_1 + \Delta w/2$, w_2, $w_2 + \Delta w/2$, $\cdots$, $w_M - \Delta w/2$, and w_M. In the block diagram, the band to be equalized extends from $w = w_1$ to $w = w_M$. [It is assumed that $E(w_1 - \Delta w/2)$ and $E(w_M + \Delta w/2)$ are zero.] It is possible that in some cases the frequencies of the oscillators could be located in the guard bands of the channel such that interference between the oscillators and the message is avoided and the process could be carried out "in service," i.e., in the presence of a "live" message load. At the equalizing station, the gradient can be generated by adding the errors with the weighting indicated in Theorem 2. Now this gradient, which is a real-time signal, is fed to the integrator the output of which can be used to adjust the corresponding gain until the optimum condition is reached with respect to MSE. In Fig. 3, the function of addition and integration is combined by using operational amplifiers.

IV. RESULTS OF COMPUTER SIMULATION

In the previous section, two algorithms were derived to obtain the MSE optimization. The general algorithm (Theorem 1) can be applied for the adjustment of a Bump Equalizer to obtain the minimum MSE, but a complex hardware implementation of this scheme may prohibit its practical application. Consequently, a simplified algorithm which is relatively simple to implement and equivalent to the general one was derived. To demonstrate the equivalence of the two algorithms, the following two conditions were assumed:

(*i*) The channel is represented by a sin x/x series in the frequency domain.

(*ii*) The Bode Network transfer function $B_k(w)$ can be approximated as in Assumption 1.

In practice, these restrictions on the channel and the equalizer are

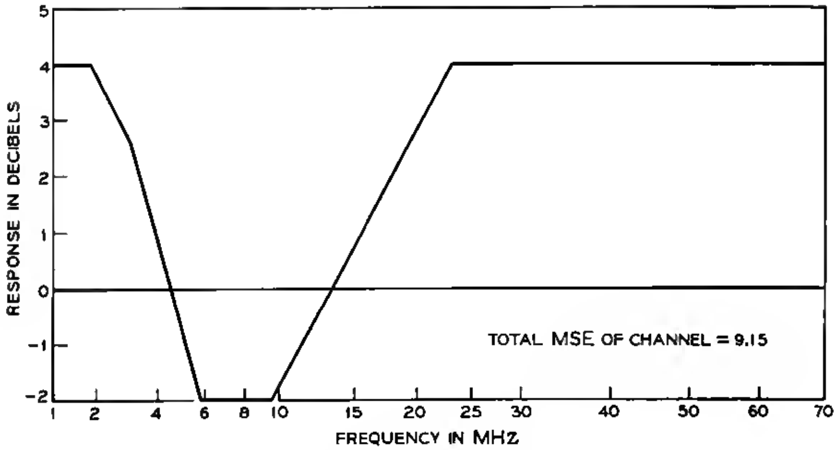

Fig. 4—Channel misalignment, case 1.

not met exactly. Bode Networks spaced equally in w, for example, do not necessarily give the best results. In fact, it has been shown experimentally that, when the number of Bode Networks is limited, unequally spaced Bode Networks are likely to achieve better equalization.[2] Hence, to verify the effectiveness of the new algorithm, the practical limitations were simulated on a digital computer, and the resulting performance compared for the two algorithms.

Case 1: The assumed channel misalignment $M(w)$ is shown in Fig. 4, over the natural frequency range from $f = 1$ MHz to $f = 65$ MHz. Transforming the lower and upper ends of the natural frequency band to the logarithmic scale, such that the message band extends from $w = 0$ ($= \log_e 1$) to $w = 4.1744$ ($= \log_e 65$), ten Bode Networks are specified and spaced equally on the w-axis. The transfer function of a physically realized Bode Network was measured and stored in the computer for this simulation.

The results of the simulation are shown in Fig. 5. The total MSE of $M(w)$ within the message band is 9.115. The values of MSE obtained by the simplified algorithm and the general algorithm are 0.45 and 0.42, respectively, with the difference resulting mainly from the following;

(*i*) Since there are sharp corners in $M(w)$, there exist some ripples whose frequency domain periods are shorter than $1/p_N$ [contradiction to the inequality (12)].

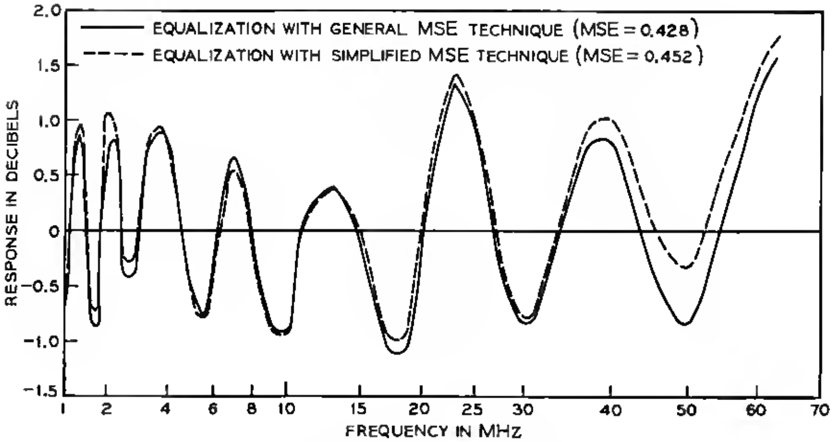

Fig. 5—Equalized channel, case 1.

(*ii*) The measured transfer function of the actual Bode Networks used in the simulation differs slightly from the cosine function of Assumption 1.

Case 2: The channel misalignment $M(w)$ used is one actually measured on an existing 20-MHz coaxial cable system, with the bandwidth

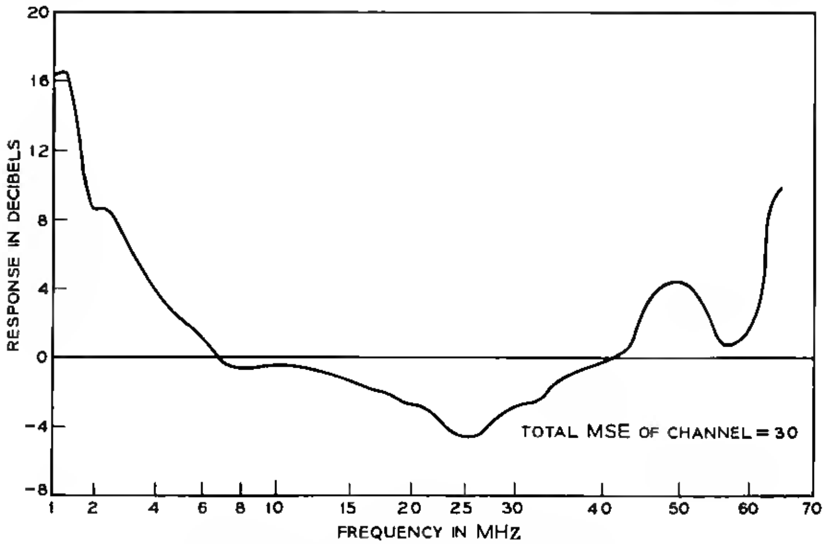

Fig. 6—Channel misalignment $M(w)$, case 2.

arbitrarily extended to 65 MHz. The misalignment $M(w)$ used in this case is shown in Fig. 6. Ten Bode Networks represented by eq. (5) are used in the equalizer, and, in this case, the center frequency of each Bode Network is initially optimized for the specified $M(w)$. Consequently, the resultant array is not spaced equally on the w-axis. To apply the simplified MSE algorithm, however, the errors are measured at 19 frequencies, 10 of which are the center frequencies of Bode Networks and 9 fall between the center frequencies. Total MSE of $M(w)$ is 30. Applying the simplified MSE algorithm, MSE = 0.201 is obtained for the equalized channel and this result is shown in Fig. 7. When the general MSE algorithm is used, the absolute minimum MSE = 0.186 is obtained. The equalized channel with the general MSE algorithm applied is also shown in Fig. 7.

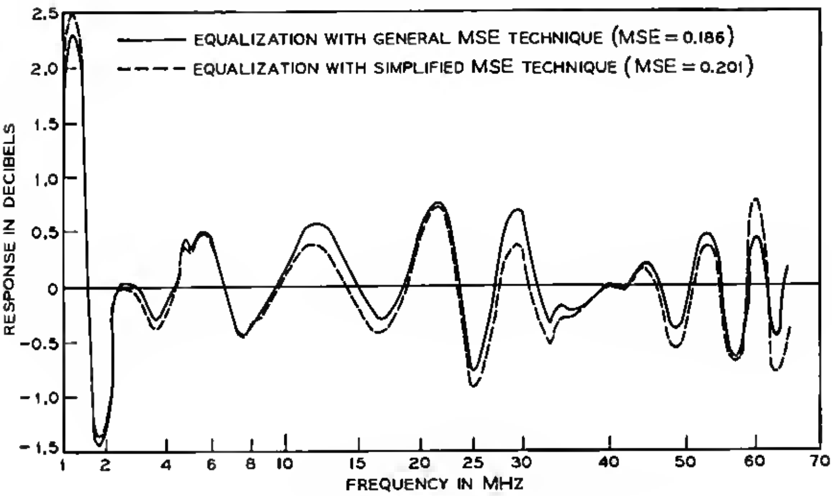

Fig. 7—Equalized channel, case 2.

V. CONCLUSIONS

Two algorithms based on the steepest descent method are presented in this paper for the optimal gain control of a Bump Equalizer. In both cases, the performance index used to evaluate the equalized channel is the MSE. The first algorithm is a general MSE algorithm and requires MSE gradient information with respect to each gain. The required gradients are obtained by a frequency domain cross-correlation between the error and the Bode Network to be adjusted. For this algorithm to be used, the error signal must be known at all

frequencies and this requirement can be a difficult one to implement physically. The simplified MSE algorithm derived in Section III, however, needs error information at only $2M - 1$ frequencies to form all of the gradients. Hence, the hardware implementation of the algorithm is more easily achieved. To derive the algorithm for the simplified case, two basic assumptions were made regarding the loss shape of the Bode Network and the characteristic of the channel. Under these assumptions, the true gradient of the kth Bode Network is given by the weighted sum of the error signal measured at frequencies w_k, $w_k + \Delta w/2$, and $w_k - \Delta w/2$, where $\Delta w = w_{k+1} - w_k$ and w_{k+1} is the center frequency of the next higher frequency Bode Network. For the hardware realization, the gradient information is applied to integrators, the outputs of which in turn control the gain settings, the process being continued until all the inputs to the integrator, i.e., gradients, become zero (see Fig. 3).

The computer results given in Section IV show that the reduction of total MSE of the equalized channel is negligible by changing from the simplified MSE algorithm to the general MSE algorithm, verifying the reasonableness of the assumptions made in Section II.

VI. ACKNOWLEDGMENT

The author wishes to thank F. C. Kelcourse with whom the author had many stimulating and encouraging discussions.

APPENDIX

Proof of Theorem 2

If we can show that the gradient

$$G_k = \Delta w \left\{ \frac{1}{2} E\left(w_k - \frac{\Delta w}{2}\right) + E(w_k) + \frac{1}{2} E\left(w_k + \frac{\Delta w}{2}\right) \right\}, \tag{15}$$

the theorem is proved by the result of Theorem 1. From eq. (10),

$$\begin{aligned} G_k &= 2 \int_{-\infty}^{\infty} B_k(w) E(w)\, dw \\ &= 2 \int_{-\infty}^{\infty} B_k(w) [\mathrm{EQL}\,(w) - M(w)]\, dw \\ &= 2(G_{k1} - G_{k2}), \end{aligned} \tag{16}$$

where

$$G_{k1} = \int_{-\infty}^{\infty} B_k(w)\, \mathrm{EQL}\,(w)\, dw$$

and

$$G_{k2} = \int_{-\infty}^{\infty} B_k(w)M(w)\,dw.$$

By Parserval's relationship,[8]

$$G_{k1} = \int_{-\infty}^{\infty} 2\pi b_k(t)\ \mathrm{eql}\,(-\mathrm{t})\,dt, \tag{17}$$

where

$$\begin{aligned} b_k(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} B_k(w)\exp\,(jwt)\,dw \\ &= \frac{\Delta w}{4\pi}\left(1+\cos\left(\frac{\Delta w}{2}\,t\right)\right)\exp\,(jw_kt) \quad \text{for} \quad |\,t\,| \leqq \frac{2\pi}{\Delta w} \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for} \quad |\,t\,| > \frac{2\pi}{\Delta w} \end{aligned}$$

and

$$\begin{aligned} \mathrm{eql}\,(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{EQL}\,(w)\exp\,(jwt)\,dw \\ &= \frac{\Delta w}{4\pi}\left(1+\cos\left(\frac{\Delta w}{2}\,t\right)\right)\sum_{i=1}^{M} g_i\exp\,(jw_it) \quad \text{for} \quad |\,t\,| \leqq \frac{2\pi}{\Delta w} \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for} \quad |\,t\,| > \frac{2\pi}{\Delta w}. \end{aligned}$$

Hence,

$$G_{k1} = \frac{\Delta w^2}{8\pi}\int_{-2\pi/\Delta w}^{2\pi/\Delta w} \sum_{i=1}^{M} g_i\exp\,(j(w_k - w_i)t)\left(1+\cos\left(\frac{\Delta w}{2}\,t\right)\right)^2 dt. \tag{18}$$

Since $w_k - w_i = (k - i)\Delta w$ and the integration with the imaginary term in (18) is zero,

$$\begin{aligned} G_{k1} &= \frac{\Delta w^2}{8\pi}\int_{-2\pi/\Delta w}^{2\pi/\Delta w} \sum_{i=1}^{M} g_i\cos\,((k-i)\,\Delta wt) \\ &\qquad \cdot\left\{\frac{3}{2} + 2\cos\left(\frac{\Delta w}{2}\,t\right) + \frac{1}{2}\cos\,(\Delta wt)\right\}dt \\ &= \frac{\Delta w^2}{8\pi}\int_{-2\pi/\Delta w}^{2\pi/\Delta w}\left\{\frac{3}{2}\,g_k + \frac{1}{4}\,g_{k-1} + \frac{1}{4}\,g_{k+1}\right\}dt \end{aligned}$$

$$= \frac{\Delta w}{2}\left\{\frac{3}{2}\, g_k + \frac{1}{4}\, g_{k-1} + \frac{1}{4}\, g_{k+1}\right\}. \tag{19}$$

However,

$$\text{EQL}\left(w_k - \frac{\Delta w}{2}\right) = \frac{1}{2}\,(g_{k-1} + g_k), \qquad \text{EQL}\,(w_k) = g_k\,,$$

and

$$\text{EQL}\left(w_k + \frac{\Delta w}{2}\right) = \frac{1}{2}\,(g_{k+1} + g_k).$$

Hence, eq. (19) becomes

$$G_{k1} = \frac{\Delta w}{2}\left[\frac{1}{2}\text{EQL}\left(w_k - \frac{\Delta w}{2}\right) + \text{EQL}\,(w_k) + \frac{1}{2}\text{EQL}\left(w_k + \frac{\Delta w}{2}\right)\right]. \tag{20}$$

Now

$$G_{k2} = \int_{-\infty}^{\infty} B_k(w) M(w)\, dw. \tag{21}$$

From eqs. (3) and (6),

$$\begin{aligned} G_{k2} = \int_{-\infty}^{\infty} &\text{cosinc}\left(\frac{\pi}{\Delta w}\,(w - w_k)\right) \int_0^1 \{F(x)\cos\,(2\pi p_N w x) \\ &+ H(x)\sin\,(2\pi p_N w x)\}\; dx\, dw \\ = \int_{-\infty}^{\infty} &\text{cosinc}\left(\frac{\pi}{\Delta w}\,(w - w_k)\right) \int_0^1 \{f(x)\cos\,(2\pi p_N (w - w_k) x) \\ &+ h(x)\sin\,(2\pi p_N (w - w_k) x)\}\; dx\, dw, \end{aligned} \tag{22}$$

where

$$F(x) = f(x)\cos\,(2\pi p_N w_k) - h(x)\sin\,(2\pi p_N w_k)$$

and

$$H(x) = f(x)\cos\,(2\pi p_N w_k) + h(x)\sin\,(2\pi p_N w_k).$$

Since cosinc $(\pi/\Delta w(w - w_k))$ and sin $(2\pi p_N(w - w_k)x)$ are even and odd functions, respectively, with respect to $w = w_k$ axis,

$$\int_0^1 \int_{-\infty}^{\infty} h(x)\;\text{cosinc}\left(\frac{\pi}{\Delta w}\,(w - w_k)\right)\sin\,(2\pi p_N (w - w_k) x)\; dw\, dx = 0.$$

Replacing $w = u + w_k$ and changing "cos" to exponential form, eq. (22) becomes

$$\frac{1}{2}\int_{-\infty}^{\infty} \text{cosinc}\left(\frac{\pi}{\Delta w}u\right)\int_0^1 f(x)(\exp(j2\pi p_N ux) + \exp(-j2\pi p_N ux))\,dx\,du$$
$$= \frac{1}{2}\int_0^1 f(x)\int_{-\infty}^{\infty} \text{cosinc}\left(\frac{\pi}{\Delta w}u\right) \cdot(\exp(j2\pi p_N ux) + \exp(-j2\pi p_N ux))\,du\,dx. \tag{23}$$

Since

$$\int_{-\infty}^{\infty} \text{cosinc}\left(\frac{\pi}{\Delta w}u\right)\exp(jut)\,du = \frac{\Delta w}{2}\left(1 + \cos\left(\frac{\Delta w}{2}t\right)\right) \quad \text{for} \quad |t| \leqq \frac{2\pi}{\Delta w}$$

and

$$= 0 \qquad \text{for} \quad |t| > \frac{2\pi}{\Delta w},$$

eq. (23) becomes

$$G_{k2} = \frac{\Delta w}{2}\int_0^1 f(x)\{1 + \cos(\pi p_N\,\Delta w x)\}\,dx \quad \text{for} \quad p_N x \leqq \frac{1}{\Delta w}$$

and

$$= 0 \qquad \text{for} \quad p_N x > \frac{1}{\Delta w}.$$

However, $0 \leqq x \leqq 1$ and $\Delta w \leqq 1/p_N$ by definition (12). Hence,

$$G_{k2} = \frac{\Delta w}{2}\int_0^1 f(x)\{1 + \cos(\pi p_N\,\Delta w x)\}\,dx$$
$$= \frac{\Delta w}{2}\left\{\frac{1}{2}M\left(w_k - \frac{\Delta w}{2}\right) + M(w_k) + \frac{1}{2}M\left(w_k + \frac{\Delta w}{2}\right)\right\}. \tag{24}$$

Combining (21) and (24), the gradient in (16) becomes

$$G_k = 2(G_{k1} - G_{k2})$$
$$= \Delta w\left\{\frac{1}{2}\left(\text{EQL}\left(w_k - \frac{\Delta w}{2}\right) - M\left(w_k - \frac{\Delta w}{2}\right)\right) + (\text{EQL}(w_k) - M(w_k)) + \frac{1}{2}\left(\text{EQL}\left(w_k + \frac{\Delta w}{2}\right) - M\left(w_k + \frac{\Delta w}{2}\right)\right)\right\}$$
$$= \Delta w\left\{\frac{1}{2}E\left(w_k - \frac{\Delta w}{2}\right) + E(w_k) + \frac{1}{2}E\left(w_k + \frac{\Delta w}{2}\right)\right\}, \tag{25}$$

which is equal to (15). This proves the theorem.

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